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HIGH-ORDER METHODS FOR DIFFERENTIAL EQUATIONS WITH LARGE FIRST-DERIVATIVE TERMS

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SUMMARY

A method is developed to solve elliptic singular perturbation problems. Examples are presented in one and two dimensions for both linear and non-linear problems. In particular, examples are presented for fluid flow problems with boundary layers. In the one-dimensional case an approximating equation is developed using just three points. The method first presented is a fourth-order approximation but is extended to become a higher-order method. Results are included for the fourth-, sixth-, eighth- and tenth-order methods.

The results are first compared with results found by Segal in an article about elliptic singular perturbation problems. The elliptic singular perturbation problems are compared with a method by Il'in and also with central and backward difference schemes from Segal's article. There was only one case where the results in Segal's paper were as accurate as the results presented in this paper. However, in this case the method used by Segal did not give accurate values for a second problem presented. The results are also compared with results given by Spalding and by Christie.

The method of this paper was also tested on the solution of some non-linear diffusion equations with concentration-dependent diffusion coefficients. The results were superior to results presented by Lee and by Schultz. Finally, the method is extended to several two-dimensional problems.

The method developed in this paper is accurate, easy to use and can be generalized to other problems.

KEY WORDS Finite differencing Iterative methods Convection–diffusion Boundary layer

1. INTRODUCTION

In this paper we have developed and tested a method for the convection–diffusion equation that can be used for small values of the viscosity coefficient. This equation is a simple model of the Navier–Stokes equations at high Reynolds numbers. The method is compared with a number of other methods.

An article by Segal[1] compared several methods for solving one- and two-dimensional problems governed by particular Navier–Stokes equations. The model in the article is a convection–diffusion equation

$$-\varepsilon\phi + \mathbf{u}\cdot\nabla\phi = f, \tag{1}$$

where ε is the viscosity coefficient (or diffusion coefficient) and $\mathbf{u}$ is the flow velocity vector.[1–13] Specific equations are described in Section 2. Most of the problems considered in the article have known solutions. Therefore the results are given as the maximum error between the numerical approximation and the exact solution.

In the one-dimensional case, this paper develops a method that uses only three points in the approximating equation. In all cases (except one where the results were comparable) it was more

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accurate than the methods described in References 1, 3, 4, 7, 13 and 14. The numerical approximations of this method can be developed as a second- or higher-order method. Section 3 includes the development of the equations needed, and results are included for the fourth-, sixth-, eighth- and tenth-order cases.

Lee[15] presented results for the problem of diffusion equations with concentration-dependent diffusion coefficients. He used a shooting technique and a fourth-order Runge–Kutta method. Schultz[14] presented a finite difference method for the same problem which was more accurate and faster than Lee's method. The method of this paper is both faster and more accurate than that of Reference 16.

The results in Section 3 are from equations given in Segal's article[1] and from Lee's[15] and Schultz's[16] papers. This allowed us to compare our results with an exact solution and also with other methods. By computing the maximum error we were able to determine which of the methods gave a better approximation. We not only obtained accurate results, but used only one method which could be generalized to all of the problems.

In particular, the method was extended to several two-dimensional problems, includng one involving a boundary layer.

Other methods proposed for similar differential equtions include those of References 3–5, 9, 10, 13 and 17.

Patankar[9] pointed out that the exponential scheme when used for the steady one-dimensional problem will give the exact solution. However, it is not widely used because exponentials are expensive to compute, the technique is problem-dependent and is not exact for two or three dimensions. The extra expense of computing the exponentials does not seem to be justified. Locally exact solutions were first employed by Allen and Southwell[17] and Spalding[13] and is one of the methods used by Raithby and Torrance.[10] However, for cell Reynolds numbers of order unity or smaller the truncation error reduces to $O(\Delta x^2, \Delta y^2)$, while for larger cell Reynolds numbers the truncation error approaches that for upstream differencing. In some cases the truncation error may be larger than for upstream differencing. The methods presented in this paper offer high-order accuracy without resorting to exponential differencing. The hybrid scheme developed by Spalding[13] is a combination of central and upwind differences. That is, it is second-order over only a part of the range. The power law difference scheme is more complicated than the hybrid scheme, but is identical to it for the absolute value of $1/\varepsilon > 10$.

Chen[5] proposed the finite-analytic method, which incorporates local analytic solutions in the numerical methods. However, this technique is very problem-dependent and he presented results only up to a Reynolds number of 10^3.

Christie and co-workers[3, 4] solved a similar problem using a high-order Galerkin method. The tenth-order scheme presented in this paper gave superior results on the same problem. Note that we could develop even higher-order methods than tenth-order for the one-dimensional problem.

2. STATEMENT OF THE PROBLEMS

In Segal's article the one-dimensional case was included to study the effect of a vertical boundary layer of width $O(\varepsilon)$ at $x = 1$. We have included three problems from the article for this case. The general form of each of these equations is

$$-\varepsilon \frac{d^2\phi}{dx^2} + u\frac{d\phi}{dx} = q,$$

with $\phi(0) = 0$ and $\phi(1) = 0$ or $\phi(1) = 1$, depending on the problem. The values for ε and u are considered to be constant, with ε defined as the viscosity coefficient and u held constant at unity.

Although ε is a constant, different values were used for each problem, varying ε from 10^{-6} to 10^{-2}. For a specific problem, q is either zero or a function of x.

In the articles by Lee[15] and Schultz[16] the problem to be considered is

$$\frac{d^2s}{dx^2} = -2xe^{s\log(1+A)}\frac{ds}{dx},$$

with $s = 1$ at $x = 0$ and $s = 0$ at $x = +\infty$. The problem is considered for various values of A.

The x-axis was subdivided into n subintervals of equal length $h = \Delta x = 1/n$, with the points labelled $x_1, x_2, x_3, \ldots, x_{n+1}$. Segal split the interval $[0, 1]$ into the intervals $[0, 1 - 8\varepsilon]$ and $[1 - 8\varepsilon, 1]$ and then subdivided each of these intervals. We did not find it necessary to do this until ε was decreased to $\varepsilon = 10^{-4}$ and $\varepsilon = 10^{-5}$.

For all of the one-dimensional problems considered it was possible to obtain approximating equations of order two or higher with just three points. The approximating equations are developed in Section 3. A separate equation was needed for the divided interval since the lengths of the subdivisions on either side of the dividing point were not equal. This equation is also included in Section 3. The generalized Newton method and the Gauss elimination method for tridiagonal systems[18-20] were used to solve the resulting systems of equtions.

The method was also extended to several two-dimensional problems. The first problem is defined by the equation

$$\phi_{xx} + \phi_{yy} - \omega\phi_x + 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y) = 0,$$

with the Dirichlet condition

$$\phi(0, y) = 0, \quad \phi(1, y) = 0, \quad \phi(x, 0) = 0, \quad \phi(x, 1) = 0.$$

The second problem is the same except for a Neumann boundary condition at $x = 1$. The exact solution is

$$\phi(x) = \sin(\pi x)\sin(\pi y).$$

These problems were also considered by Segal.

The third problem was considered by Strikwerda[21] and has a boundary layer. The problem is defined by the equation

$$\phi_{xx} + \phi_{yy} + \omega\phi_x = 0,$$

with the boundary conditions

$$\phi(0, y) = y(1 - y), \quad \phi(1, y) = \left(y(1 - y) - \frac{2}{\omega}\right)e^{-\omega}, \quad \phi(x, 0) = -\frac{2x}{\omega}e^{-\omega x},$$

$$\phi(x, 1) = -\frac{2x}{\omega}e^{-\omega x}.$$

The exact solution is

$$\phi(x, y) = \left(y(1 - y) - \frac{2x}{\omega}\right)e^{-\omega x}.$$

3. METHOD

The first case to be considered is

$$-\varepsilon\phi''(x) + \phi'(x) = 0, \tag{2}$$

with boundary conditions

$$\phi(0) = 0, \quad \phi(1) = 1{\cdot}0. \tag{3}$$

3.1. Development of a three-point, fourth-order method

To develop the equations for the fourth-order method, equation (2) is rewritten in the form

$$\phi''(x) - \omega\phi'(x) = 0, \tag{4}$$

where $\omega = 1/\varepsilon$.

The interval [0, 1] is divided into n equal subdivisions, each of length $h = 1/n$. This gives $n-1$ function values to be found in the interval. The differential equation will be approximated using three points arranged on the x-axis as in Figure 1. Point 0 corresponds to x_i.

At each x_i we want to find α_0, α_1 and α_3 such that

$$\phi''(x) - \omega\phi'(x) \approx \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_3\phi_3. \tag{5}$$

We use the Taylor series to expand about the point ϕ_0 to obtain

$$\begin{aligned}\phi''(x) - \omega\phi'(x) \approx \alpha_0\phi_0 &+ \alpha_1\Big(\phi_0 + h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) + \frac{h^3}{6}\phi'''(x_0) \\ &+ \frac{h^4}{24}\phi^{(4)}(x_0) + \frac{h^5}{120}\phi^{(5)}(x_0) + \cdots\Big) \\ &+ \alpha_3\Big(\phi_0 - h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) - \frac{h^3}{6}\phi'''(x_0) + \frac{h^4}{24}\phi^{(4)}(x_0) \\ &- \frac{h^5}{120}\phi^{(5)}\,\phi^{(5)}(x_0) + \cdots\Big)\end{aligned} \tag{6}$$

We rewrite equation (4), take its derivative twice and get the derivatives for $\phi'''(x)$ and $\phi^{(4)}(x)$ in terms of $\phi''(x)$. This gives

$$\begin{aligned}\phi''(x) &= \omega\phi'(x), \\ \phi'''(x) &= \omega\phi''(x), \\ \phi^{(4)}(x) &= \omega^2\phi''(x).\end{aligned} \tag{7}$$

Substituting (7) into (6) gives

$$\begin{aligned}\phi''(x) - \omega\phi'(x) \approx \alpha_0\phi_0 &+ \alpha_1\Big(\phi_0 + h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) + \frac{h^3}{6}\omega\phi''(x_0) \\ &+ \frac{h^4}{24}\Big(\omega^2\phi''(x_0)) + \frac{h^5}{120}\phi^{(5)}(x_0) + \cdots\Big) \\ &+ \alpha_3\Big(\phi_0 - h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) - \frac{h^3}{6}\omega\phi''(x_0) \\ &+ \frac{h^4}{24}\omega^2\phi''(x_0) - \frac{h^5}{120}\phi^{(5)}(x_0) + \cdots\Big).\end{aligned} \tag{8}$$

Setting corresponding coefficients equal gives the following system of equations:

$$\alpha_0 + \alpha_1 + \alpha_3 = 0, \tag{9}$$

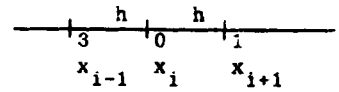

Figure 1. Arrangement of approximating points

$$h(\alpha_1 - \alpha_3) = -\omega, \tag{10}$$

$$\alpha_1\left(\frac{h^2}{2} + \frac{h^3}{6}\omega + \frac{h^4}{24}\omega^2\right) + \alpha_3\left(\frac{h^2}{2} - \frac{h^3}{6}\omega + \frac{h^4}{24}\omega^2\right) = 1. \tag{11}$$

This system is solved to find the values of the α_i:

$$\begin{aligned} \alpha_3 &= \frac{24 + 12h\omega + 4h^2\omega^2 + h^3\omega^3}{24h^2 + 2h^4\omega^2}, \\ \alpha_1 &= \alpha_3 - \omega/h, \\ \alpha_0 &= -\alpha_1 - \alpha_3. \end{aligned} \tag{12}$$

The error term comes from the Taylor series expansion in (6) and is given by

$$\frac{h^5}{120}\phi^{(5)}(x_0)(\alpha_1 - \alpha_3) = \frac{h^5}{120}\phi^{(5)}(x_0)\left(\alpha_3 - \frac{\omega}{h} - \alpha_3\right) = -\frac{\omega}{120}h^4\phi^{(5)}(x_0), \tag{13}$$

which is $O(h^4)$. We now have a fourth-order method using just three points.

3.2. Higher-order methods

To obtain a higher-order method, we add more terms to the Taylor series expansion in (6). It is necessary to add two terms to get a sixth-order method, four terms for an eighth-order method, six terms for a tenth-order method, etc. Since the order of the method depends on the two error terms in (13) having opposite signs, the order will always be an even number.

The Taylor series expansion for a sixth-order method is carried out to the seventh-derivative term. We also need the values found for the derivatives in (7) along with

$$\begin{aligned} \phi^{(5)}(x_0) &= \omega^2\phi'''(x_0) = \omega^3\phi''(x_0), \\ \phi^{(6)}(x_0) &= \omega^3\phi'''(x_0) = \omega^4\phi''(x_0) \end{aligned} \tag{14}$$

to replace the derivatives of order three or greater in the Taylor series. Solving the system as before, we get α_0 and α_1 as in (12) and

$$\alpha_3 = \frac{720 + 360\omega h + 120\omega^2h^2 + 30\omega^3h^3 + 6\omega^4h^4 + \omega^5h^5}{2(360h^2 + 30h^4\omega^2 + h^6\omega^4)}. \tag{15}$$

The error term is found from the Taylor series expansion and is given by

$$-\frac{\omega}{5040}h^6\phi^{(7)}(x). \tag{16}$$

This method was also extended to eighth- and tenth-order methods by simply adding more terms to the Taylor series expansion about ϕ_0 and solving the system of equations for α_3, since α_0

and α_1 do not change. The coefficient α_3 is given by

$$\alpha_3 = \frac{40\,320 + 20\,160\omega h + 6720\omega^2 h^2 + 1680\omega^3 h^3 + 336\omega^4 h^4 + 56\omega^5 h^5 + 8\omega^6 h^6 + \omega^7 h^7}{40\,320h^2 + 3360h^4\omega^2 + 112h^6\omega^4 + 2\omega^6 h^8}$$

for the eighth-order method and

$$\alpha_3 = AB,$$

where

$$A = \frac{1\,814\,400}{1\,814\,400h^2 + 151\,200h^4\omega^2 + 5040h^6\omega^4 + 90h^8\omega^6 + h^{10}\omega^8},$$

$$B = 1 + \frac{\omega}{h}\left(\frac{h^2}{2} + \omega\frac{h^3}{6} + \omega^2\frac{h^4}{24} + \omega^3\frac{h^5}{120} + \omega^4\frac{h^6}{720} + \omega^5\frac{h^7}{5040} + \omega^6\frac{h^8}{40\,320} + \omega^7\frac{h^9}{362\,880} + \omega^8\frac{h^{10}}{3\,628\,800}\right),$$

for the tenth-order method.

3.3. Equation for the dividing point

As ω in equation (4) was increased (or ε was decreased), it was necessary to use a divided interval to obtain accurate results. The left side of the interval, $[0, 1 - 8\varepsilon]$, was divided into n subdivisions, each of length

$$k = (1 - 8\varepsilon)/n.$$

The right side of the interval, $[1 - 8\varepsilon, 1]$, was divided into m subdivisions, each of length

$$h = 8\varepsilon/m.$$

Since the original equation was developed for equal subdivisions and this is no longer true at the dividing point, a new equation was developed for this point.

The equation used at the dividing point for the fourth-order case was also developed using three points as before. The point were arranged as in Figure 2.

For the fourth-order method the Taylor series expansion for the dividing point was carried out to the seventh-derivative term in order to obtain a more accurate solution. With equal subdivisions this number of terms in the Taylor series expansion gave a sixth-order method in Section 3.2. With unequal subdivisions, however, the error term in (18) contains α_3, which does not drop out as it did in (16), and the error is not as small. Therefore more terms were used to increase the accuracy.

The following results were obtained for the dividing point:

$$\alpha_0 = -\alpha_1 - \alpha_3,$$

$$\alpha_1 = \alpha_3\frac{k}{h} - \frac{\omega}{h}, \tag{17}$$

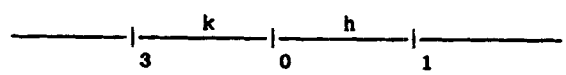

Figure 2. Arrangement of points at the dividing point

$$\alpha_3 = \frac{720 + 360\omega h + 120h^2\omega^2 + 30h^3\omega^3 + 6h^4\omega^4 + h^5\omega^5}{360hk + 120h^2k\omega + 30h^3k\omega^2 + 6h^4k\omega^3 + h^5k\omega^4 + 360k^2 - 120k^3\omega + 30k^4\omega^2 - 6k^5\omega^3 + k^6\omega^4}.$$

The largest error term is

$$\phi^{(7)}(x_0)\frac{1}{5040}[\alpha_3 k(h^6 - k^6) - \omega h^6]. \tag{18}$$

3.4. *Results*

Segal[1] divided the interval into the two subdivisions

$$[0, 1 - 8\varepsilon] \quad \text{and} \quad [1 - 8\varepsilon, 1].$$

Each of these subdivisions used $(n + 1)/2$ points and equal spacing. The methods used were a central difference scheme, a backward difference scheme and the Il'in method.[14] The results are given for the two most accurate methods, the central difference scheme and the Il'in method, for $n = 21$ and $n = 41$ in Table I. See Figure 3 for a comparison of the methods of this paper with the central difference scheme. Figure 4 compares the results of the three different schemes with the exact solution using only three points ($h = 1/2$). It corresponds to Figure 4 in Spalding's paper.[13]

We checked our method by first using a uniform grid spacing of $h = 1/n$. As ε decreased, we found we could obtain better results if we split the interval into two subintervals, $[0, 1 - 8\varepsilon]$ and $[1 - 8\varepsilon, 1]$.

Table I. Maximum error for central difference and Il'in methods from Segal's paper with $(n + 1)/2$ nodes for each subinterval

Method	n	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
Central	21	0·021	0·021	0·021	0·021
difference	41	0·005	0·005	0·005	0·005
Il'in	21	10^{-7}	10^{-15}	10^{-15}	10^{-15}
	41	10^{-5}	10^{-15}	10^{-15}	10^{-15}

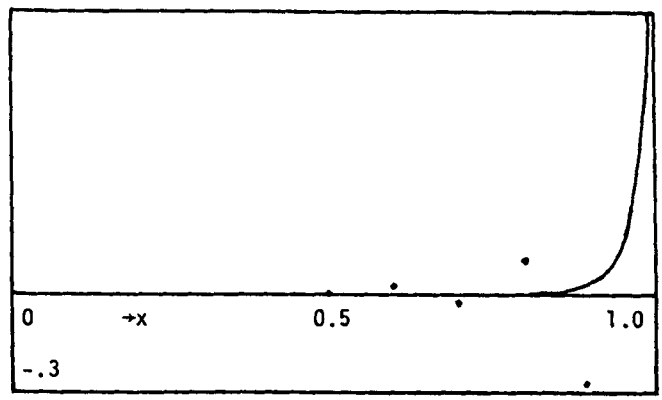

Figure 3. The curved line is the exact solution of problem 1. The results of the methods of this paper (sixth, eighth and tenth order) fall exactly on this curve. The dots are the corresponding results from the central difference method. $N = 10$, $\omega = 1/\varepsilon = 40$

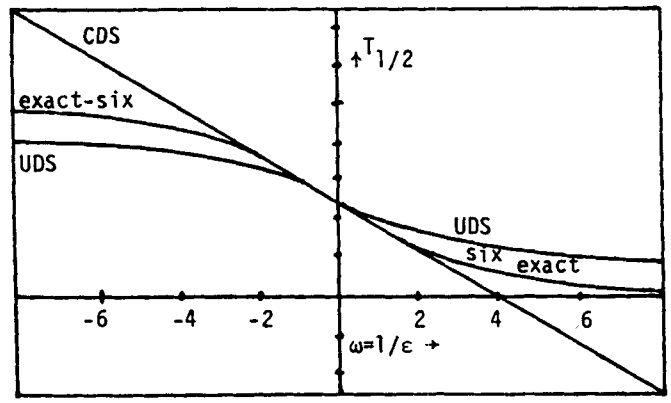

Figure 4. Dependence of the solution at the midpoint according to the exact solution (problem 1) and to three finite difference methods. SIX is the sixth-order method of this paper. CDS is the central difference method. UDS is the upward difference method. Note exact and SIX solutions coincide. $h = 1/2$ (only three points used)

Table II. Maximum error for fourth-order method, problem 1

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n, m	Maximum error	n, m	Maximum error
40	0·05						
80	0·005	500	0·02				
125[a]	0·0008	1250	0·0008	1000,10	0·0008	1000,10	NR
250[b]	0·00005	2500	0·00005	1000,20	0·0001	1000,20	0·0004
						1000,40	0·0003

n is the number of subdivisions.
n, m corresponds to n subdivisions in the interval $[0, 1 - 8\varepsilon]$ and m subdivisions in the interval $[1 - 8\varepsilon, 1]$.
[a] Row corresponds to $n = 21$ in Segal's results in Table I.
[b] Row corresponds to $n = 41$ in Segal's results in Table I.
NR: no results.

The maximum error for each subdivision is summarized in Tables II, III, IV and V for methods of order four, six, eight and ten respectively. Since the exact value is known for these problems, the maximum error is the absolute value of the difference between the approximate and exact values, or

$$\text{maximum error} = \max|\text{approximate value} - \text{exact value}|.$$

Even though the number of subdivisions had to be increased to obtain the results in Tables II–V, the number of iterations did not become a prohibitive number. In Table II, with 2500 subdivisions and $\varepsilon = 10^{-3}$, only 67 iterations were required for convergence. For the eighth-order method in Table IV, 351 iterations were required for convergence of 25 000 subdivisions with $\varepsilon = 10^{-4}$. In Table V, 25 000 subdivisions for $\varepsilon = 10^{-4}$ required 427 iterations for convergence. This huge number of subdivisions was used only to show the stability of the method and the ease of solving the large system.

To compare the accuracy for each of the methods given in Tables II–V, consider the case when $\varepsilon = 10^{-2}$. For the fourth-order method the maximum error for $n = 250$ is given in Table II as

Table III. Maximum error for sixth-order method, problem 1

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n, m	Maximum error	n, m	Maximum error
		500	0·003				
125[a]	0·00001	1250	0·00001	3000,10	0·00005	3000,10	NR
250[b]	0·000001	2500	0·0000003	3000,20	0·00004	3000,20	0·0003

[a, b] See footnotes to Table II.

Table IV. Maximum error for eighth-order method, problem 1

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n	Maximum error	n, m	Maximum error
125[a]	0·0000003	1250	0·0000003	12500	0·000002	3000,10	0·008
250[b]	0·0000001	2500	0·0000001	25000	0·0000003	3000,20	0·000006

[a, b] See footnotes to Table II.

Table V. Maximum error for tenth-order method, problem 1

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n or n, m	Maximum error	n or n, m	Maximum error	n or n, m	Maximum error	n, m	Maximum error
20	0·005	500	0·000005	1000	$0{\cdot}7 \times 10^{-8}$	3000, 10	0·000004
40	0·00003	1000	$0{\cdot}8 \times 10^{-8}$	4000, 100	$0{\cdot}2 \times 10^{-6}$		
80	$0{\cdot}5 \times 10^{-6}$	2000	$0{\cdot}7 \times 10^{-11}$	5000, 100	$0{\cdot}2 \times 10^{-7}$		
100	$0{\cdot}8 \times 10^{-8}$	5000	$0{\cdot}1 \times 10^{-11}$	6900, 100	$0{\cdot}7 \times 10^{-9}$		
500	$0{\cdot}2 \times 10^{-13}$	1000, 100	$0{\cdot}15 \times 10^{-10}$	10000, 100	$0{\cdot}2 \times 10^{-11}$		
20, 20	$0{\cdot}1 \times 10^{-3}$	4000, 100	$0{\cdot}2 \times 10^{-12}$				
40, 40	$0{\cdot}8 \times 10^{-7}$						
100, 100	$0{\cdot}7 \times 10^{-11}$						
1000, 100	$0{\cdot}2 \times 10^{-13}$						

5×10^{-5}. The sixth-order method already has an error of 10^{-5} for $n = 125$ subdivisions. In Table IV the error is 10^{-7} for 125 subdivisions, and for the tenth-order method the error is 5×10^{-7} for 80 subdivisions and 10^{-13} for 500 subdivisions..

It was found that better results could be obtained if we did not divide the region into the two subregions $[0, 1 - 8\varepsilon]$ and $[1 - 8\varepsilon, 1]$, but instead used the subregions [0, 0·999] and [0·999, 1·0].

The results are given in Table VI. The equations were solved by the direct method for tridiagonal systems. Note that we had no trouble solving the system even up to 10 000 equations.

The problem was also run on the IMSL routine DVCPR. This routine solves boundary value problems using a variable order method with deferred corrections. The method worked only for $\varepsilon = 0{\cdot}01$ and $\varepsilon = 0{\cdot}001$. It would not work for smaller ε. The best result for $\varepsilon = 0{\cdot}001$ was a maximum error of $0{\cdot}4 \times 10^{-5}$ with 300 points.

Table VI. Maximum error using the subregions [0, 0·999], [0·999, 1·0] and the tenth-order method

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
n, m	Maximum error	n, m	Maximum error	n, m	Maximum error	n, m	Maximum error	n, m	Maximum error
40, 40	$0{\cdot}4 \times 10^{-4}$					1000, 1000	$0{\cdot}7 \times 10^{-10}$		
1000, 40	$0{\cdot}1 \times 10^{-13}$	1000, 40	$0{\cdot}1 \times 10^{-7}$	1000, 40	0·05				
4000, 100	$0{\cdot}1 \times 10^{-13}$	4000, 100	$0{\cdot}1 \times 10^{-12}$	4000, 100	$0{\cdot}3 \times 10^{-7}$	4000, 100	$0{\cdot}8 \times 10^{-8}$		
4000, 1000	$0{\cdot}8 \times 10^{-13}$	4000, 1000	$0{\cdot}8 \times 10^{-12}$	4000, 1000	$0{\cdot}3 \times 10^{-7}$	4000, 1000	$0{\cdot}8 \times 10^{-10}$	4000, 1000	$0{\cdot}7 \times 10^{-8}$
		9000, 1000	$0{\cdot}8 \times 10^{-12}$	9000, 1000	$0{\cdot}8 \times 10^{-11}$				

3.5. Development of the fourth-order equation for the second problem

The second problem to be considered is

$$-\varepsilon\phi''(x) + \phi'(x) = \varepsilon\pi^2 \sin(\pi x) + \pi \cos(\pi x), \tag{19}$$

with $$\phi(0) = 0, \quad \phi(1) = 1.$$

The equation was rewritten as

$$\phi''(x) - \omega\phi'(x) + \pi^2 \sin(\pi x) + \omega\pi \cos(\pi x) = 0. \tag{20}$$

This equation is again approximated by three points, arranged as in Figure 1. To obtain high-order methods for this problem, we had to include a term α_4 (see equation (21)).

The Taylor series expansion is again used to approximate the left-hand side of (20) as follows:

$$\begin{aligned}\phi''(x) - \omega\phi'(x) + \pi^2 \sin(\pi x) + \omega\pi \cos(\pi x) &\approx \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_3\phi_3 + \alpha_4 \\ &= \alpha_0\phi_0 + \alpha_1\left(\phi_0 + h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) + \frac{h^3}{6}\phi'''(x_0) + \frac{h^4}{24}\phi^{(4)}(x_0) + \frac{h^5}{120}\phi^{(5)}(x_0) + \cdots\right) \\ &\quad + \alpha_3\left(\phi_0 - h\phi'(x_0) + \frac{h^2}{2}\phi''(x_0) - \frac{h^3}{6}\phi'''(x_0) + \frac{h^4}{24}\phi^{(4)}(x_0) - \frac{h^5}{120}\phi^{(5)}(x_0) + \cdots\right) + \alpha_4.\end{aligned} \tag{21}$$

When $\phi'''(x)$ and $\phi^{(4)}(x)$ are written in terms of $\phi''(x)$ we obtain the approximating equation as before.

The values for α_0, α_1 and α_3 are the same as in (12), and α_4 is given as

$$\begin{aligned}\alpha_4 = &-\alpha_3\left(\frac{h^4}{12}[\omega^2\pi^2 \sin(\pi x) + \pi^4 \sin(\pi x)]\right) + \frac{h^2}{6}[-\pi^3\omega \cos(\pi x) + \omega^2\pi^2 \sin(\pi x)] \\ &+ \frac{h^3}{24}[\pi^2\omega^3 \sin(\pi x) + \omega\pi^4 \sin(\pi x)] + \pi^2 \sin(\pi x) + \omega\pi \cos(\pi x).\end{aligned} \tag{22}$$

The error term is the same as in (13), which makes this method $O(h^4)$.

3.6. Higher-order methods for the second problem

The higher-order methods are obtained in a way similar to that of Section 3.2. That is, two more terms are needed to obtain a sixth-order method, four terms for an eighth-order method, etc. The sixth-order values for α_0, α_1 and α_3 are the same for the second problem as those for the first. The difference here is that the term α_4 must be calculated.

When simplified, the sixth-order value for α_4 is

$$\begin{aligned}\alpha_4 = &-\alpha_3\left(\frac{h^4}{12}[\omega^2\pi^2 \sin(\pi x) + \pi^4 \sin(\pi x)] + \frac{h^6}{360}[\omega^4\pi^2 \sin(\pi x) - \pi^6 \sin(\pi x)]\right) \\ &+ \frac{\omega}{h}\left(\frac{h^3}{6}[-\pi^3 \cos(\pi x) + \omega\pi^2 \sin(\pi x)] + \frac{h^4}{24}[\omega^2\pi^2 \sin(\pi x) + \pi^4 \sin(\pi x)]\right. \\ &\left. + \frac{h^5}{120}[\omega^3\pi^2 \sin(\pi x) + \pi^5 \cos(\pi x)] + \frac{h^6}{720}[\omega^4\pi^2 \sin(\pi x) - \pi^6 \sin(\pi x)]\right) \\ &+ \pi^2 \sin(\pi x) + \omega\pi \cos(\pi x).\end{aligned} \tag{23}$$

The approximating equation then is

$$\phi''(x) - \omega\phi'(x) + \pi^2 \sin(\pi x) + \omega\pi \cos(\pi x) \approx \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_3\phi_3 + \alpha_4.$$

Even though it may appear that α_4 will become too difficult to work with as the order of the method increases, it is actually only a matter of adding terms to the α_4 of the previous order. Thus we can theoretically develop an arbitrarily high-order method. Eighth- and tenth-order methods were also developed.

3.7. Results for the second problem

Segal again used an interval divided as before with $(n + 1)/2$ points in each interval and equal spacing. The same three methods were used: backward difference, central difference and Il'in method.

From Table VII it can be seen that the central difference method $O(h^2)$ gave similar errors as for problem 1. However, the errors for the Il'in scheme, which had been very accurate for the first problem, were closed to those of the backward scheme, which is an $O(h)$ method.

The maximum error for the method from Sections 3.5 and 3.6 is summarized in Tables VIII–XI for fourth, sixth, eighth and tenth orders.

For $\varepsilon = 10^{-2}$ to 10^{-4} the results in Tables VIII–XI are significantly better than Segal's in Table VII for comparable subdivisions. The eighth- and tenth-order methods gave errors as small as $0{\cdot}2 \times 10^{-13}$ as compared to 0·005 in Table VII. For $\varepsilon = 10^{-5}$ in all methods, however, it was necessary to use a very small overrelaxation factor of 0·0125 and initial values from the $\varepsilon = 10^{-4}$ run in order to obtain a converged solution. Better results were then obtained by using the direct method for a tridiagonal system (Table XI).

Note that for $n = 100$ and $\varepsilon = 10^{-2}$ the central approximation had an error of 0·034 while our tenth-order method had a maximum error of 0·000 000 1. For $\varepsilon = 10^{-4}$ the best result for the

Table VII. Segal's maximum error from Reference 1

Method	n	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
Central	21	0·022	0·021	0·021	0·021
	41	0·005	0·005	0·005	0·005
Backward	21	0·279	0·305	0·309	0·309
	41	0·140	0·155	0·156	0·156
Il'in	21	0·212	0·299	0·308	0·309
	41	0·081	0·149	0·156	0·156

Table VIII. Maximum error for fourth-order method, problem 2

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n or n, m	Maximum error	n, m	Maximum error	n, m	Maximum error
125[a]	0·0008	1250	0·0006	3000, 10	0·0009		
250[b]	0·00005	500, 20	0·00008	3000, 20	0·0002	3000, 20	0·002
						3000, 40	0·002

[a, b] See footnotes to Table II.

Table IX. Maximum error for sixth-order method, problem 2

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n, m	Maximum error	n, m	Maximum error
80	0·0002	500	0·003				
125[a]	0·00001	1250	0·00002	3000, 10	0·00006	3000, 10	NR
250[b]	0·000002	2500	0·00001	3000, 20	0·00004	3000, 20	0·002
						3000, 40	0·002

[a, b] See footnotes to Table II.

Table X. Maximum error for eighth-order method, problem 2

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n, m	Maximum error	n, m	Maximum error
80	0·000005	500	0·0002				
125[a]	0·0000005	1250	0·000001	3000, 10	0·000006	3000, 10	NR
250[b]	0·0000002	2500	0·0000005	3000, 20	0·000005	3000, 20	NR
						3000, 40	0·005

[a, b] See footnotes to Table II.

Table XI. Maximum error for tenth-order method, problem 2

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n or n, m	Maximum error	n or n, m	Maximum error	n or n, m	Maximum error	n, m or n	Maximum error
80	$0{\cdot}1 \times 10^{-6}$	400	0·00004	2000	0·006	3000,10	NR
200	$0{\cdot}7 \times 10^{-11}$	500	0·000006	5000	0·000006	25000	0·0095
1000	$0{\cdot}1 \times 10^{-13}$	750	$0{\cdot}1 \times 10^{-6}$	7000	$0{\cdot}2 \times 10^{-6}$	45000	0·0007
20, 20	0·0001	1000	$0{\cdot}7 \times 10^{-8}$	10000	$0{\cdot}7 \times 10^{-8}$		
100, 100	$0{\cdot}7 \times 10^{-11}$	10000	$0{\cdot}15 \times 10^{-12}$	3000, 10	0·000005		
1000, 100	$0{\cdot}2 \times 10^{-13}$	1000, 100	$0{\cdot}15 \times 10^{-10}$	4000, 10	$0{\cdot}2 \times 10^{-6}$		
		4000, 100	$0{\cdot}2 \times 10^{-12}$				

central approximation was a maximum error of 0·004 for $n = 40\,000$ while our method had an error of 0·000 000 2 for $n = 7000$.

It was again found that better results could be obtained if we did not divide the region into the two subregions $[0, 1 - 8\varepsilon]$ and $[1 - 8\varepsilon, 1]$, but instead used the subgregions [0, 0·999] and [0·999, 1·0].

The results are given in Table XII. Note that even for $\varepsilon = 10^{-6}$ we were able to obtain a maximum error of $0{\cdot}1 \times 10^{-6}$. For $\varepsilon = 10^{-2}$ the best maximum error was $0{\cdot}1 \times 10^{-13}$.

Table XII. Maximum error using the subregions [0, 0·999], [1·0] and the methods of this paper

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-6}$	
n, m	Maximum error	n, m	Maximum error	n, m	Maximum error	n, m	Maximum error	n, m	Maximum error
50, 50	$0{\cdot}5 \times 10^{-5}$	1000, 100	$0{\cdot}1 \times 10^{-7}$	4000, 100	$0{\cdot}3 \times 10^{-7}$	400, 500	$0{\cdot}2 \times 10^{-3}$	4000, 1000	$0{\cdot}7 \times 10^{-3}$
100, 100	$0{\cdot}8 \times 10^{-8}$	4000, 100	$0{\cdot}1 \times 10^{-12}$	9000, 1000	$0{\cdot}8 \times 10^{-11}$	500, 50	$0{\cdot}3 \times 10^{-4}$	8000, 2000	$0{\cdot}3 \times 10^{-5}$
1000, 100	$0{\cdot}1 \times 10^{-13}$					1000, 100	$0{\cdot}6 \times 10^{-7}$	9000, 1000	$0{\cdot}1 \times 10^{-6}$
						4000, 100	$0{\cdot}8 \times 10^{-8}$		
						4000, 1000	$0{\cdot}8 \times 10^{-10}$		

3.8. Problem 3

The third problem in the one-dimensional case is given by

$$-\varepsilon\phi''(x) + \phi'(x) = \varepsilon\pi^2 \sin(\pi x) + \pi \cos(\pi x) \tag{24}$$

with boundary conditions

$$\phi(0) = 0, \quad \phi(1) = 0.$$

Equation (24) is the same equation as (19) in Section 3.5, but the boundary conditions are different.

Segal included this case, when there is no normal boundary layer, to consider whether the oscillations that occurred in his results for the previous two problems were due to the presence of a normal boundary layer or to the fact that the matrix was not always diagonally dominant. We also had oscillations occur for coarse grid sizes, which affected our results in several cases, but were able to avoid them by going to a smaller grid size. Although no table was included in Segal's paper for this case, a graph was given which showed the central difference method $O(h^2)$ to be more accurate than the Il'in scheme. In fact, Segal notes that the Il'in scheme 'tried to generate a boundary layer'.

To approximate the points in the interval for this problem, the same equation (21) was used as for problem 2 with the same values for the α_i given by (12) and (22). Results are included for the fourth- and sixth-order methods in Tables XIII and XIV.

As can be seen by comparing Table XIII with Table VIII and Table XIV with Table IX, the results for problem 3 are generally better than those of problem 2. Also, very accurate results were obtained for problem 3 using a smaller number of subdivisions than was possible in problem 2. For example, in Table XIV the error for $\varepsilon = 10^{-2}$ using 80 subdivisions is 10^{-8} but in Table IX for the same case the error is 0·0002. Also of interest is the error column for $\varepsilon = 10^{-5}$ in Table XIV. The results for this case are very accurate without using a divided interval. For $\varepsilon = 10^{-5}$ in problem 2 we had to use a divided interval to obtain a converged solution and the maximum error was $0{\cdot}8 \times 10^{-10}$ in Table XII.

Table XIII. Maximum error for fourth-order method, problem 3

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n	Maximum error	n, m	Maximum error
40	$0{\cdot}3 \times 10^{-6}$	250	$0{\cdot}2 \times 10^{-6}$	500	0·000001	3000, 10	0·002
80	$0{\cdot}2 \times 10^{-7}$	500	$0{\cdot}4 \times 10^{-7}$	1250	$0{\cdot}2 \times 10^{-8}$	3000, 20	0·002
						3000, 40	0·002

Table XIV. Maximum error for sixth-order method, problem 3

$\varepsilon = 10^{-2}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-5}$	
n	Maximum error	n	Maximum error	n	Maximum error	n	Maximum error
40	$0{\cdot}4 \times 10^{-7}$	250	$0{\cdot}1 \times 10^{-7}$	500	$0{\cdot}1 \times 10^{-7}$	2500	$0{\cdot}5 \times 10^{-6}$
80	$0{\cdot}1 \times 10^{-7}$			1250	$0{\cdot}2 \times 10^{-8}$	5000	$0{\cdot}3 \times 10^{-7}$

Note that a standard five-point fourth-order method was also applied to these problems. The method was harder to use and the results were not as accurate as our fourth-order method. The defect correction method[1, 7, 22] was also used on this problem. The results were similar to those of the central difference approximation and not nearly as good as the method of this paper.

3.9. Problem 4

The method described in this paper was also applied to the problem of solving some diffusion equations with concentration-dependent diffusion coefficients, which had been considered previously by Lee[15] and later by Schultz.[16] The method described in Reference 16 was more accurate and up to ten times faster than Lee's method. The method described in this paper was found to be superior to both.

The problem to be considered is

$$\frac{d^2s}{dx^2} = -2xe^{s\log(1+A)}\frac{ds}{dx},$$

with $s = 1$ at $x = 0$ and $s = 0$ at $x = +\infty$. The condition at infinity can be handled as in Reference 16.

This equation describes the problem of diffusion during the uptake of excess calcium by calcium fluoride. The solution is obtained for various values of A, where A is the critical concentration of trapped electrons.

The method is developed similarly to the previous problems.

Let $\alpha = \log(1 + A)$ and write the equation in the form

$$y'' = -2xe^{\alpha y}y'.$$

y''' and $y^{(4)}$ are developed from this equation and can be simplified to the following form:

$$y''' = \frac{y''}{x} - \frac{\alpha(y'')^2}{2xe^{\alpha y}} - \alpha xe^{\alpha y}y'',$$

$$y^{(4)} = \frac{-\alpha(y'')^2}{x^2 - e^{\alpha y}} - 6e^{\alpha y}y'' + \frac{\alpha^2(y'')^3}{(2xe^{\alpha y})^2} + 4(\alpha y'')^2 + (2xe^{\alpha y})^2 y''.$$

The equation is then approximated using only three points by

$$y'' + 2xe^{\alpha y}y' = \alpha_0 y_0 + \alpha_1 y_1 + \alpha_3 y_3.$$

Expanding y_1 and y_3 in Taylor series about zero, as before, we obtain the equations

$$\alpha_0 + \alpha_1 + \alpha_3 = 0,$$

$$h\alpha_1 - h\alpha_3 = 2xe^{\alpha y},$$

$$\alpha_3\left(h^2 - \frac{h^4\alpha y''}{12x^2e^{\alpha y}} - \frac{h^4e^{\alpha y}}{3} + \frac{h^4\alpha^2(y'')^2}{48x^2(e^{\alpha y})^2} + \frac{h^4\alpha y''}{4} - \frac{h^4e^{\alpha y}}{6} + \frac{h^4\alpha y''}{12} + \frac{h^4x^2(e^{\alpha y})^2}{3}\right)$$

$$= 1 - \left(hxe^{\alpha y} + \frac{h^2e^{\alpha y}}{3} - \frac{h^2\alpha y''}{6} - \frac{2x^2h^2(e^{\alpha y})^2}{3} - \frac{h^3\alpha y''}{12x} - \frac{h^3x(e^{\alpha y})^2}{3}\right.$$

$$\left. + \frac{h^3\alpha^2(y'')^2}{48xe^{\alpha y}} + \frac{h^3\alpha xe^{\alpha y}}{4}y'' - \frac{h^3(e^{\alpha y})^2x}{6} + \frac{h^3\alpha xe^{\alpha y}}{12}y'' + \frac{h^3x^3(e^{\alpha y})^3}{3}\right).$$

It can easily be shown that this gives an $O(h^4)$ method if we approximate y'' in the normal way.

Table XV. Effects of h on diffusion problem from Reference 16 for $A = 0{\cdot}5$

h	$x = 0{\cdot}1$	$x = 2{\cdot}2$
0·1	0·87852	0·001536
0·05	0·87876	0·001615
0·025	0·87882	0·001635
0·0125	0·87883	0·001640
0·01	0·87883	0·001640

Table XVI. Effects of h on diffusion problem using (fourth-order) method of this paper for $A = 0{\cdot}5$

h	$x = 0{\cdot}1$	$x = 2{\cdot}2$
0·2		0·001630
0·1	0·87884	0·001640
0·05	0·87884	0·001640

Table XV gives the results from Reference 16 and Table XVI shows the results from our method. From the tables we can see that the results from our method with $h = 0{\cdot}1$ are equivalent to the results for $h = 0{\cdot}0125$ given in Reference 16.

4. EXTRAPOLATION

For the one-dimensional case the error for the fourth-order method was

$$E = ch^4,$$

where c is considered constant. Let I_h be the approximate value obtained by the fourth-order method using h as the step size and let I_k be the approximate value obtained using k as the step size. Then an improved value I can be obtained by[18, 19]

$$I = I_h + \frac{I_h - I_k}{k^4/h^4 - 1}. \tag{25}$$

For problem 1 an example was chosen from each of the values for $\varepsilon = 10^{-5}$ to 10^{-2} for the fourth-order method. The maximum error of 0·0008 for $\varepsilon = 10^{-2}$ with 125 subdivisions occurred when $x = 0{\cdot}992$. The value for I_k was chosen for 250 subdivisions when $x = 0{\cdot}992$. The error here was 0·000 05. Using equation (25), a new approximation I was found with an error of 0·000 002, which is comparable to the error for the sixth-order method with 250 subdivisions. The results are summarized in Table XVII. Similar improvements could be made for the six-, eighth- and tenth-order methods.

5. METHOD FOR TWO-DIMENSIONAL PROBLEMS

The method is based on a nine-point approximation using a Taylor series expansion about the point ϕ_0 which represents the function value at the point (x_i, y_i). Figure 5 shows the placement of

Table XVII. Extrapolation results for the fourth-order method from Section 3, problem 1

	n or n, m	x	Error	Error for I
ε^{-2}	125	0·992	0·0008	0·000002
	250	0·992	0·00005	
ε^{-3}	1250	0·9992	0·0008	0·000002
	2500	0·9992	0·00005	
ε^{-4}	1000, 20	0·99992	0·0001	0·000004
	1000, 40	0·99992	0·0003	
ε^{-5}	1000, 20	0·999996	0·00012	0·000008
	1000, 40		0·00018	

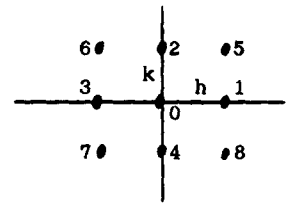

Figure 5. Placement of points for two-dimensional problems

the nine points. The differential equation is

$$\phi_{xx} + \phi_{yy} - \omega\phi_x + 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y) = 0,$$

with the Dirichlet conditions

$$\phi(0, y) = 0, \quad \phi(1, y) = 0, \quad \phi(x, 0) = 0, \quad \phi(x, 1) = 0.$$

The subdivision in the x-direction is $h = 1/n$ and in the y-direction is $k = 1/m$, where n and m are the numbers of subdivisions in the x- and y-direction respectively.

Since the differential equation has the following function of x and y,

$$2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y),$$

on the left-hand side, a term α_9 was included to represent a function of x and y in the approximation. At each point (x_i, y_i) we want to find the values $\alpha_0, \alpha_1, \cdots, \alpha_9$ such that the approximating equation is of the form

$$\phi_{xx} + \phi_{yy} - \omega\phi_x + 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y) \approx \sum_0^8 \alpha_i\phi_i + \alpha_9. \tag{26}$$

The values for the unknown coefficients $\alpha_0, \alpha_1, \ldots, \alpha_9$ are again found by Taylor series expansion about the point ϕ_0:

$$\phi_{xx} + \phi_{yy} - \omega\phi_x + 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y) \approx \sum_0^8 \alpha_i\phi_i + \alpha_9 \tag{27}$$

$$= \phi_0(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8) \tag{28}$$

$$+ \phi_{x_0}(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8)h \tag{29}$$

$$+ \phi_{y_0}(\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8)k \tag{30}$$

$$+ \phi_{xx_0}(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)h^2/2 \tag{31}$$

$$+ \phi_{xy_0}(\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8)hk \tag{32}$$

$$+ \phi_{yy_0}(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)k^2/2 \tag{33}$$

$$+ \phi_{xxx_0}(\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8)h^3/6 \tag{34}$$

$$+ \phi_{xxy_0}(\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8)kh^2/2 \tag{35}$$

$$+ \phi_{xyy_0}(\alpha_5 - \alpha_6 - \alpha_7 + \alpha_8)hk^2/2 \tag{36}$$

$$+ \phi_{yyy_0}(\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8)k^3/6 \tag{37}$$

$$+ \phi_{xxxx_0}(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)h^4/24 \tag{38}$$

$$+ \phi_{yyyy_0}(\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)k^4/24 \tag{39}$$

$$+ \phi_{xxxy_0}(\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8)h^3k/6 \tag{40}$$

$$+ \phi_{xxyy_0}(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)h^2k^2/4 \tag{41}$$

$$+ \phi_{xyyy_0}(\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8)hk^3/6 \tag{42}$$

$$+ \alpha_9. \tag{43}$$

If we set the coefficients in terms (28)–(43) equal to the corresponding coefficients on the left side of equation (26), we get the following system of equations:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0, \tag{44}$$

$$\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = -\omega/h, \tag{45}$$

$$\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0, \tag{46}$$

$$\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 2/h^2, \tag{47}$$

$$\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0, \tag{48}$$

$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 2/k^2, \tag{49}$$

$$\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0, \tag{50}$$

$$\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0, \tag{51}$$

$$\alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = 0, \tag{52}$$

$$\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0, \tag{53}$$

$$\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0, \tag{54}$$

$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0, \tag{55}$$

$$\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0, \tag{56}$$

$$\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0, \tag{57}$$

$$\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0, \tag{58}$$

$$\alpha_9 = 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega \cos(\pi x)\sin(\pi y). \tag{59}$$

Equations (46) and (53) are duplicate equations so (53) will be left out. The same is true for equations (48), (56) and (58). Therefore equations (56) and (58) are not needed. Also, equations (45)

and (50) cannot be true at the same time so we must go back and rewrite the Taylor series expansion. The same is true for equations (47) and (54) and again for equations (49) and (55). We will do this by rewriting the partial derivatives in the expressions (34), (38) and (39), which are the corresponding expressions for (50), (54) and (55) respectively. Therefore a replacement is needed for the partial derivatives ϕ_{xxx}, ϕ_{xxxx} and ϕ_{yyyy}

By first rewriting the differential equations as

$$\phi_{xx} = \omega\phi_x - \phi_{yy} - 2\pi^2 \sin(\pi x)\sin(\pi y) - \pi\omega\cos(\pi x)\sin(\pi y), \tag{60}$$

it is possible to obtain the partials

$$\phi_{xxx} = \omega\phi_{xx} - \phi_{xyy} - 2\pi^3\cos(\pi x)\sin(\pi y) + \pi^2\omega\sin(\pi x)\sin(\pi y), \tag{61}$$

$$\phi_{xxxx} = \omega\phi_{xxx} - \phi_{xxyy} + 2\pi^4\sin(\pi x)\sin(\pi y) + \pi^3\omega\cos(\pi x)\sin(\pi y),$$

which become

$$\phi_{xxxx} = \omega^2\phi_{xx} - \omega\phi_{xyy} - 2\omega\pi^3\cos(\pi x)\sin(\pi y) + \pi^2\omega^2\sin(\pi x)\sin(\pi y) - \phi_{xxyy} + 2\pi^4\sin(\pi x)\sin(\pi y) + \pi^3\omega\cos(\pi x)\sin(\pi y) \tag{62}$$

when ϕ_{xxx} is replaced by its equivalent from (61).

By rewriting the differential equation as

$$\phi_{yy} = \omega\phi_x - \phi_{xx} - 2\pi^2\sin(\pi x)\sin(\pi y) - \pi\omega\cos(\pi x)\sin(\pi y)$$

and taking the partial derivatives with respect to y twice, we obtain

$$\phi_{yyyy} = \omega\phi_{xyy} - \phi_{xxyy} + 2\pi^4\sin(\pi x)\sin(\pi y) + \pi^3\omega\cos(\pi x)\sin(\pi y). \tag{63}$$

If we substitute the partial derivatives obtained in (61), (62) and (63) into expressions (34), (38) and (39) and then equate corresponding coefficients with (26), we obtain the following system of ten equations and ten unknowns:

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = 0, \tag{64}$$

$$\alpha_1 - \alpha_3 + \alpha_5 - \alpha_6 - \alpha_7 + \alpha_8 = -\omega/h, \tag{65}$$

$$\alpha_2 - \alpha_4 + \alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0, \tag{66}$$

$$(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\left(\frac{h^2}{2} + \omega^2\frac{h^4}{24}\right) - \omega^2\frac{h^2}{6} = 1, \tag{67}$$

$$\alpha_5 - \alpha_6 + \alpha_7 - \alpha_8 = 0, \tag{68}$$

$$\alpha_2 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = \frac{2}{k^2}, \tag{69}$$

$$\alpha_5 + \alpha_6 - \alpha_7 - \alpha_8 = 0, \tag{70}$$

$$(\alpha_5 - \alpha_6 - \alpha_7 + \alpha_8)\left(\frac{hk^2}{2}\right) + \omega\frac{h^2}{6} - \omega(\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\frac{h^4}{24} + \omega\frac{k^2}{12} = 0, \tag{71}$$

$$(\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)h^2\frac{k^2}{4} - \frac{k^2}{12} - (\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\frac{h^4}{24} = 0, \tag{72}$$

$$\alpha_9 + [-2\pi^3\cos(\pi x)\sin(\pi y) + \pi^2\omega\sin(\pi x)\sin(\pi y)]\left(-\frac{\omega}{h}\right)\left(\frac{h^3}{6}\right)$$

$$+ [-2\omega\pi^3 \cos(\pi x)\sin(\pi y) + \pi^2\omega^2 \sin(\pi x)\sin(\pi y) + 2\pi^4 \sin(\pi x)\sin(\pi y)$$

$$+ \pi^3\omega \cos(\pi x)\sin(\pi y)](\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8)\frac{h^4}{24} + [2\pi^4 \sin(\pi x)\sin(\pi y)$$

$$+ \pi^3\omega \cos(\pi x)\sin(\pi y)]\frac{k^2}{12} = 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega \cos(\pi x)\sin(\pi y). \tag{73}$$

The following are the steps used to solve the system of equations in (64)–(73).

(a) Combine (68) with (70) giving $\alpha_5 = \alpha_8$.
(b) Equation (65) minus (68) gives $\alpha_1 - \alpha_3 = 2\alpha_7 - 2\alpha_8 - \omega/h$.
(c) Equation (65) minus (70) gives $\alpha_1 - \alpha_3 = 2\alpha_6 - 2\alpha_8 - \omega/h$.
(d) Set equation (b) equal to (c) giving $\alpha_7 = \alpha_6$.
(e) Equation (67) can be written as

$$\alpha_1 + \alpha_3 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = \frac{4(6 + \omega^2h^2)}{12h^2 + \omega^2h^4}$$

(f) Substitute equation (e) in (72).
(g) Substitute equations (a) and (d) in (f) giving an expression for $\alpha_5 + \alpha_6$.
(h) Equation (66) plus (69) gives $\alpha_2 + \alpha_5 + \alpha_6 = 1/k^2$.
(i) Substitute equation (g) in (h) giving an expression for α_2.
(j) Equations (a) and (d) in (66) gives $\alpha_4 = \alpha_2$.
(k) Equations (a), (d) and (e) in (71) solves for $\alpha_5 - \alpha_6$.
(l) Equation (g) plus (k) solves for α_5.
(m) The remaining unknown values for the α_i can now be found in terms of the values already found in steps (a)–(l).

Since the values $\alpha_0, \alpha_1, \ldots, \alpha_9$ can be complicated, no attempt was made to simplify these expressions. All were used as listed below in the computer program. The coefficients for the approximating equation in (26) are

$$\begin{aligned}
\alpha_0 &= -\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7 - \alpha_8,\\
\alpha_1 &= -2\alpha_8 + \frac{2(6 + \omega^2h^2)}{12h^2 + \omega^2h^4} - \frac{\omega}{2h},\\
\alpha_2 &= -\frac{1}{6h^2} - \frac{6 + \omega^2h^2}{3k^2(12 + \omega^2h^2)} + \frac{1}{k^2},\\
\alpha_3 &= \alpha_1 - 2\alpha_6 + 2\alpha_8 + \omega/h,\\
\alpha_4 &= \alpha_2,\\
\alpha_5 &= \frac{1}{2h^2k^2}\left(\frac{k^2}{6} - \omega\frac{h^3}{6} - \omega h\frac{k^2}{12} + \frac{h^2(6 + \omega^2h^2)(2 + \omega h)}{6(12 + \omega^2h^2)}\right),\\
\alpha_6 &= -\alpha_5 + \frac{1}{h^2k^2}\left(\frac{k^2}{6} + h^2\frac{(6 + \omega^2h^2)}{3(12 + \omega^2h^2)}\right),\\
\alpha_7 &= \alpha_6,\\
\alpha_8 &= \alpha_5,
\end{aligned} \tag{74}$$

$$\alpha_9 = 2\pi^2 \sin(\pi x)\sin(\pi y) + \pi\omega\cos(\pi x)\sin(\pi y) + \omega\frac{h^2}{6}[-2\pi^3\cos(\pi x)\sin(\pi y)$$
$$+ \pi^2\omega\sin(\pi x)\sin(\pi y)] + [2\omega\pi^3\cos(\pi x)\sin(\pi y) - \pi^2\omega^2\sin(\pi x)\sin(\pi y)$$
$$- 2\pi^4\sin(\pi x)\sin(\pi y) - \pi^3\omega\cos(\pi x)\sin(\pi y)]\left(\frac{6+\omega^2h^2}{12+\omega^2h^2}\right)\frac{h^2}{6} - [2\pi^4\sin(\pi x)\sin(\pi y)$$
$$+ \pi^3\omega\cos(\pi x)\sin(\pi y)]\frac{k^2}{12}.$$

The error term for the approximation is

$$E = -\frac{\omega}{120}h^4\phi_{xxxxx}.$$

5.1. Approximating equation for Neumann boundary condition

The second problem in the two-dimensional case has a Neumann boundary condition at $x = 1$ of $\phi_x = \pi\cos(\pi x)\sin(\pi y)$. Therefore a new equation must be developed for the nodes when $x = 1$.

The points to be used are arranged as in Figure 6. The approximation used is

$$\phi_x \simeq \alpha_0\phi_0 + \alpha_1\phi_1 + \alpha_2\phi_2 + \alpha_3\phi_3 + \alpha_4\phi_4,$$

where

$$\alpha_0 = \frac{25}{12h}, \quad \alpha_1 = -\frac{4}{h}, \quad \alpha_2 = \frac{3}{h}, \quad \alpha_3 = -\frac{4}{3h}, \quad \alpha_4 = \frac{1}{4h}.$$

The error term is determined from the Taylor series expansion as

$$E = -\tfrac{1}{5}h^4\phi_{xxxxx}.$$

The results for central differences are given in Table XVIII and the results from the method of this paper are given in Table XIX. Note that the results of this paper are considerably better than the central difference results.

For the second problem with a Neumann boundary condition when $x = 1$, Segal chose equal subdivisions on the x- and y-axis of $n = 10$ and $n = 20$. Table XX gives the results using the

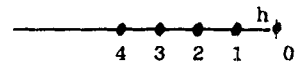

Figure 6. Arrangement of points for Neumann boundary condition at $x = 1$

Table XVIII. Maximum error for central difference (Dirichlet boundary conditions)

$n \times n$	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
5 × 5	0·03	0·03	0·03	0·03
10 × 10	0·008	0·008	0·008	0·008
20 × 20	0·002	0·002	0·002	0·002
40 × 40	0·0005	0·0005	0·0005	0·0005

central difference scheme, upwind difference scheme and defect correction method. The most accurate of the three methods for this problem was the central difference scheme.

Equal subdivisions in the x- and y-direction also gave accurate results for the second problem using the equations developed in Section 5. In Table XXI the maximum error is shown for n subdivisions of 10, 20 and 40.

Our maximum errors were constant from problem 1 to problem 2. The results for $n = 10$ and $\varepsilon = 10^{-2}$, 10^{-3} and 10^{-4} are better than Segal's central difference errors by a factor of over 50, while our results for $n = 20$ are better by a factor of 200.

For the last problem[21] the equation is

$$\phi_{xx} + \phi_{yy} + \omega\phi_x = 0,$$

with

$$\phi(0, y) = y(1 - y), \quad \phi(1, y) = \left(y(1 - y) - \frac{2}{\omega} \right) e^{-\omega}, \quad \phi(x, 0) = \frac{-2x}{\omega} e^{-\omega x} = \phi(x, 1).$$

Table XIX. Maximum error for problem 1 with Dirichlet boundary conditions and n subdivisions in the x- and y-direction

n	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
10	0·0003	0·0003	0·0003	0·0003
20	0·00002	0·00002	0·00002	0·00002
40	0·000002	0·000002	0·000002	0·000002

Table XX. Segal's maximum errors for Neumann boundary condition at $x = 1$, problem 2

Method	n	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
Central difference	10	0·017	0·017	0·017
	20	0·004	0·004	0·004
Upwind difference	10	0·306	0·309	0·309
	20	0·154	0·156	0·156
Defect correction	10	0·080	0·081	0·081
	20	0·039	0·040	0·040

Table XXI. Maximum error for problem 2 with Neumann boundary condition at $x = 1$

n	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$
10	0·003	0·003	0·003	0·003
20	0·00002	0·00002	0·00002	0·00002
40	0·000002	0·000002	0·000002	0·000002

Note that this problem has a boundary layer. The approximating equation is similar to the last problem. The method is stable and converged rapidly for all values of ω attempted. The results were again much better than the central difference results, which converged only for small ω. As an example, for $\omega = 10$ and a grid of 10×20 the central differences had a maximum error of $0{\cdot}2 \times 10^{-2}$ while we had a maximum error of only $0{\cdot}3 \times 10^{-4}$. See Figures 7 and 8 for a comparison of the fourth-order method and the central difference technique for $\omega = 40$ and a grid of 20×20. Note that the exact results and the fourth-order results coincide.

6. RESULTS

Our results were more accurate for all problems tested, except for one case where the results were comparable. The first problem in the one-dimensional case was the only problem where Segal's results were comparable to ours. Our best case gave an error of 10^{-14} in a tenth-order method and Segal's best error was 10^{-15} using the Il'in method. However, for the second and third

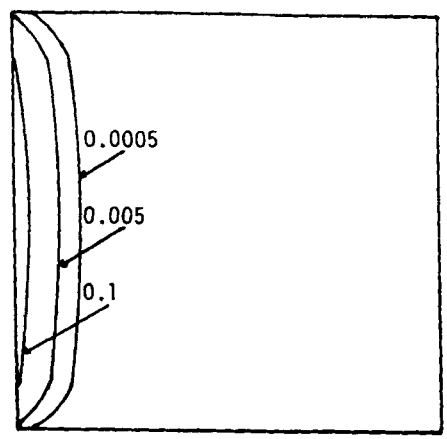

Figure 7. Problem 3, exact and fourth-order results, two dimensions, 20×20 grid, $\omega = 1/\varepsilon = 40$. The results of the fourth-order method and the exact reslts coincide. Most of the results are close to zero

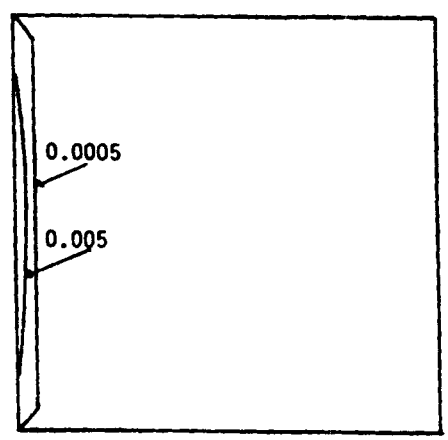

Figure 8. Problem 3, two dimensions, 20×20 grid, $\omega = 1/\varepsilon = 40$, central differences

problems the Il'in method gave very poor results and the central difference scheme $O(h^2)$ had a maximum error of 0·005 for problem 2. For comparable subdivisions (marked with [a] and [b] in the tables) with $\varepsilon = 10^{-4}$ to 10^{-2}, our worst result for problem 2 using the fourth-order method was 0·0009 while our best result was considerably better, 0·000 05. Also, we were able to improve these results by using a higher-order method and going to more subdivisions. The results given in Table XII are considerably better than Segal's results in Table VII.

7. DIFFICULTY OF METHOD

One consideration in the use of any method might be phrased: 'How difficult is it to apply?' For the simplest problem, the equation was given in (4) as

$$\phi''(x) - \omega\phi'(x) = 0.$$

No constant term was needed in the Taylor series expansion and the resulting system of three equations with three unknowns was easily solved for the fourth-order method. To increase the order of the method, more terms were included in the Taylor series expansion, but the values of α_0 and α_1 were the same as those found for the fourth-order approximating equation. The value of α_3 was the only one to change. This change was simply a matter of including more terms in α_3 than used for the previous method.

When a constant term α_4 had to be included in the development of the approximating equation for (20), the values for α_0, α_1 and α_3 were no different than those for problem 1. The constant α_4 was long but easily written in terms of α_3. Again, to increase the order of the method, the values of α_0, α_1 and α_3 did not change and α_4 was found by adding terms to the Taylor series expansion. The expression became lengthy but not difficult to find or express in terms of α_3.

8. CONCLUSIONS

We have developed a numerical method that can be applied to both one-dimensional and two-dimensional fluid flow problems governed by Navier–Stokes equations. The order of the two-dimensional method is $O(h^4)$. For the one-dimensional problem, results were included for fourth-, sixth-, eighth- and tenth-order methods.

The results of the method were accurate. Only in one case was Segal's results comparable to ours. However, from Section 4 we see that extrapolation improves our results. Also, the method used by Segal did not give accurate answers for the second problem in the one-dimensional case. Our results were accurate for all problems considered from $\varepsilon = 10^{-5}$ to 10^{-2} and the method could be generalized to both one- and two-dimensional problems. This was not true for any of the other methods tested.

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